

Valence Parton Distribution Functions from Quantum ChromoDynamics

S. G. Rajeev*

*Department of Physics and Astronomy
University of Rochester, Rochester, New York 14627*

We show that two dimensional QCD can, to a good approximation, describe the hadronic structure functions measured in Deep Inelastic Scattering. We transform this theory into a new form, Quantum HadronDynamics (QHD) , whose semi-classical approximation is closer to nature. The Baryon is then a topological soliton, and its structure function can be predicted by a variational principle. This prediction can be tested by comparison with measurements of neutrino scattering cross-sections.

Keywords: Structure Functions; Parton Model; Deep Inelastic Scattering; Neutrino Scattering; QCD; Skyrme model; Quantum HadronDynamics.

PACS : 12.39Ki,13.60.-r, 12.39Dc,12.38Aw.

Perturbative QCD allows us to determine the Q^2 dependence of the structure functions of Deep Inelastic Scattering [1]. The initial condition for the evolution of the DGLAP equations [2](i.e., the dependence on the Bjorken scaling variable x_B at an initial Q^2) cannot be determined within the perturbative framework. As a result much effort has been expended to extract these initial distributions by fits to data (See e.g., [3].) We describe in this paper how to derive the valence quark distributions of the proton (in particular the dependence on x_B at a low initial Q^2) from first principles of QCD, by a succession of approximations. It is possible to test the predictions against experimental data: a comparison with the measurement of the structure function xF_3 in neutrino scattering will be done in a companion paper [4]. The agreement is quite good, which confirms the theoretical framework advocated in this paper. There are also some other approaches to this problem which are more numerical in character [5]

As we will explain below, Deep Inelastic Scattering can be explained by the dimensional reduction of QCD to two spacetime dimensions. The main idea is now to rewrite two dimensional QCD in terms of operators that describe mesons rather than quarks and gluons [6]. This new formulation which I called Quantum HadronDynamics (QHD) has the advantage that its semi-classical approximation is quite close to nature: it corresponds to the large N_c limit of QCD. Recall that the semi-classical approximation to QCD itself is invalid except at short distances. In particular it fails to explain the formation of hadrons.

In QHD, the baryon appears as a topological soliton. Its structure functions are determined by a variational principle, within the large N_c limit. There is a natural variational ansatz which corresponds to the valence quark approximation. Within this ansatz we can even take care of the leading effect of N_c being finite: it just amounts to restricting the range of momenta allowed for partons. Thus we will be able to obtain a variational principle for the valence quark distribution functions. A more detailed version of this argument can be found in [8].

*rajeev@pas.rochester.edu

Let us begin by recalling why Deep Inelastic Scattering can be understood within a two dimensional framework [1]. An electron (or neutrino) scatters against a proton (or other nucleus) with a space like momentum transfer of magnitude q , causing the target to disintegrate into many hadrons. The two Lorentz invariant variables that describe this process can be chosen to be $Q^2 = -q^2$ and the Bjorken variable $x_B = -\frac{q^2}{2p \cdot q}$. It is easy to check that $0 < x_B < 1$. Deep inelastic scattering is the limiting case $Q^2 \rightarrow \infty$ keeping x_B fixed. More precisely $Q \gg a^{-1}$ where a is the characteristic size of the target. In this limit, in the center of mass frame the target will look ‘flattened out’ to a pancake shape due to Lorentz contraction. To first approximation, it can be thought of as having infinite extent in the two tranverse directions while of finite extent a in the longitudinal direction. In other words the momenta of the constituents of the hadron can be taken to be zero in the transverse direction, the corresponding fields are then independent of the transverse spatial co-ordinates. Thus the theory of strong interactions (QCD) can be dimensionally reduced to $1+1$ space-time dimensions in describing deep inelastic scattering. We will use the methods of Ref. [6] to study this two dimensional field theory. This will give us the structure function as a function of x_B .

The effects of the transverse momenta can then be included as a perturbative correction. As emphasized by Altarelli and Parisi [2] this is precisely the meaning of the DGLAP evolution equations, which give the dependence of the structure functions on Q^2 . (The upper cut-off on the allowed transverse momentum is Q , which is how it is related to transverse momentum effects.) Since this part of the story is standard [7] we will not discuss it here.

In the above two dimensional approximation, the transversely polarized gluons will appear as a pair of scalar fields. Since we don’t need the gluon structure functions for now, we can ignore these transverse gluons. To leading order, the evolution of the valence parton distributions decouple from the gluon distribution functions. The longitudinal component of the gluon cannot be ignored as it is responsible for the binding of the quarks into the hadron. However, they don’t have dynamical degrees of freedom and can be eliminated using their equations of motion.

Thus the action of the two dimensional field theory describing the Deep Inelastic Structure function is,

$$S = \frac{N_c}{4g^2} \int \text{tr} F_{\mu\nu} F^{\mu\nu} d^2x + \sum_{a=1}^{2N_f} \int \bar{q}^{a\alpha} [-i\gamma \cdot \nabla + m_a] q_{a\alpha} d^2x. \quad (1)$$

Here, $\alpha = 1, \dots, N_c$ is the color index and $a = 1, \dots, 2N_f$ a flavor index. Also, q is a two dimensional Dirac spinor ; each four dimensional spinor splits into a pair of these upon dimensional reduction and hence the two dimensional theory has twice the number of flavors as the four dimensional theory. In null co-ordinates (for more about the kinematics in the null co-ordinates, see [8]) ,and in the light-cone gauge $A_- = 0$, we can eliminate the remaining gluon degrees of freedom to get the hamiltonian

$$H = \int dx \chi^{\dagger ai} \frac{1}{2} [\hat{p} + \frac{m^2}{\hat{p}}] \chi_{ai} - \frac{1}{2N} \alpha_1 \int \frac{1}{2} |x-y| : \chi^{\dagger ai}(x) \chi_{aj}(x) :: \chi^{\dagger bj}(y) \chi_{bi}(y) : dx dy \quad (2)$$

where $q = \begin{pmatrix} q_1 \\ \frac{1}{\sqrt{2}}\chi \end{pmatrix}$. The upper component q_1 is not a propagating degree of freedom and has been eliminated in terms of χ . The quark field χ satisfies the canonical anti-commutation relations

$[\chi(x), \chi(y)]_+ = \delta(x - y)$, $[\chi(x), \chi^\dagger(y)]_+ = 0$. Also, the normal ordered product $:AB:$ is defined with respect to the vacuum $\tilde{\chi}^\dagger(p)|0\rangle = 0$ for $p < 0$, $\tilde{\chi}(p)|0\rangle = 0$ for $p > 0$.

Now define the color invariant variable $\hat{M}_b^a(x, y) = \frac{1}{N_c} : [\chi_{ba}(x), \chi^{\dagger aa}(y)] :$ which can be thought of as the field operator for a meson field. The space-time points x, y lie along a null line which is thought of as the initial value surface.

Now the entire theory can be described in terms of this color invariant variable. Within the subspace of color invariant states, $\hat{M}_b^a(x, y)$ is a complete set of observables: the only operators that commute with them are multiples of the identity. This follows from the fact that $\hat{M}(x, y)$ provide an irreducible (projective) unitary representation of the infinite dimensional unitary Lie algebra:

$$\{\tilde{M}_b^a(p, q), \tilde{M}_d^c(r, s)\} = \frac{1}{N_c} \left(\delta_b^c 2\pi \delta(q - r) [\delta_d^a \text{sgn}(p - s) + \tilde{M}_d^a(p, s)] - \delta_d^a 2\pi \delta(s - p) [\delta_b^c \text{sgn}(r - q) + \tilde{M}_b^c(r, q)] \right). \quad (3)$$

Here $\tilde{M}_b^a(p, q) = \int \hat{M}_b^a(x, y) e^{ipx - iquy} dx dy$. Note that the commutators are of order $\frac{1}{N_c}$ so that the large N_c limit is a sort of classical limit: $\frac{1}{N_c}$ plays the role of \hbar in an ordinary field theory.

In this classical limit the above commutators tend to Poisson brackets of a set of classical dynamical variables $M_b^a(x, y)$. The phase space of this classical dynamical system must be a homogenous symplectic manifold, this being the analogue of an irreducible unitary representation. From the standard Kirillov theory (adapted to infinite dimensions by Segal, [9]) this phase space is a co-adjoint orbit of the unitary group, the Grassmannian. It is the set of all infinite dimensional operators M with integral kernel $M_b^a(x, y)$ satisfying the nonlinear constraint $[\epsilon + M]^2 = 1$. Here, ϵ is the Hilbert transform operator, $\epsilon(x, y) = \int e^{ip(x-y)} \text{sgn}(p) \frac{dp}{2\pi}$. It is also possible to verify the identity above directly on color singlet states as is shown in the appendix to Ref. [11]. Thus in the large N_c limit, our problem reduces to solving the equations of motion obtained from the hamiltonian

$$E[M] = -\frac{1}{4} \int [p + \frac{\mu_a^2}{p}] \tilde{M}(p, p) \frac{dp}{2\pi} + \frac{\tilde{g}^2}{8} \int M_b^a(x, y) M_a^b(y, x) |x - y| dx dy \quad (4)$$

with the Poisson brackets

$$\frac{1}{2i} \{M_b^a(x, y), M_d^c(z, u)\} = \delta_b^c \delta(y - z) [\epsilon_d^a(x, u) + M_d^a(x, u)] - \delta_d^a \delta(x - u) [\epsilon_b^c(z, y) + M_b^c(z, y)]. \quad (5)$$

The parameter μ_a^2 is related to the quark masses by a finite renormalization: $\mu_a^2 = m_a^2 - \frac{\tilde{g}^2}{\pi}$, where m_a is the *current* quark mass. (Also, $\tilde{M}_b^a(p, q) = \int M_b^a(x, y) e^{-ipx + iquy} dx dy$ is the Fourier transform.)

What kind of solution to this theory represents the baryon? The quantity $B = -\frac{1}{2} \int M_a^a(x, x) dx$ can be shown to be an integer, a topological invariant [6]. From the definition of M in terms of χ, χ^\dagger we can see that this is in fact the baryon number. Thus the baryon is a topological soliton in this picture: an idea originally proposed by Skyrme in quite a different context, and revived by Balachandran et. al. and by Witten et. al. [10]. We seek a static solution (minimum of the energy subject to constraints) that has baryon number one. Again from the definition in terms of the quark fields, we can see that $-\frac{1}{2} \tilde{M}_a^a(p, p)$ represents the quark number density in momentum space, when $p > 0$. Similarly, $\frac{1}{2} \tilde{M}_a^a(-p, -p)$ represents the anti-quark number density. It is convenient to assume a variational ansatz of the separable (rank one) form $\tilde{M}_b^a(p, q) = -2\tilde{\psi}^a(p)\tilde{\psi}_b^*(q)$. This satisfies the constraint if $\tilde{\psi}$ is of norm one and of positive momentum: $\sum_a \int_0^\infty |\tilde{\psi}_a(p)|^2 \frac{dp}{2\pi} = 1$, $\tilde{\psi}_a(p) = 0$, for $p < 0$. This variational satisfies the Poisson bracket relations $\{\tilde{\psi}_a(p), \tilde{\psi}_b(q)\} = 0$, $\{\tilde{\psi}_a(p), \tilde{\psi}^{*b}(q)\} = -i2\pi \delta_a^b \delta(p - q)$. The energy becomes then,

$$E_1(\psi) = \sum_a \int_0^\infty \frac{1}{2} \left[p + \frac{\mu^2}{p} \right] |\tilde{\psi}_a(p)|^2 \frac{dp}{2\pi} + \frac{\tilde{g}^2}{2} \sum_{ab} \int \int |\psi_a(x)|^2 |\psi_b(y)|^2 \frac{1}{2} |x - y| dx dy. \quad (6)$$

This variational ansatz corresponds to the valence quark approximation: the anti-quark distributions are identically zero. In forthcoming papers with V. John and G. S. Krishnaswami, we will show that in the limit of zero current quark mass the *exact* minimum of the energy functional is of this separable form [4,12]. In fact deviations are small even for finite m ; in the language of the parton model, there is a less than one percent probability of finding an anti-quark in the proton at low Q^2 . The factorized ansatz amounts to ignoring the anti-quarks.

So far we have worked in the large N_c limit. What is the first order effect of N_c being finite? If we stay within the valence parton approximation as we make N_c finite this can be studied by replacing the Poisson brackets of ψ by canonical commutation relations:

$$[\hat{\psi}_a(p), \hat{\psi}_b(p')] = 0 = [\hat{\psi}^{\dagger a}(p), \hat{\psi}^{\dagger b}(p')], \quad [\hat{\psi}_a(p), \hat{\psi}^{\dagger b}(p')] = \frac{1}{N_c} 2\pi \delta(p - p') \delta_a^b. \quad (7)$$

As usual, classical Poisson brackets go over to quantum commutation relations, except that the role of \hbar is played by $\frac{1}{N_c}$. These commutation relations have a simple representation in terms of bosonic creation-annihilation operators. The constraint on ψ then becomes the condition that there be exactly N_c such bosons in any allowed state. The large N_c limit is then like a thermodynamic limit, in the canonical ensemble.

What are these bosons? A moment's reflection will show that they are in fact the valence partons: we have just given a derivation of the valence parton model from a series of approximations on QCD. They behave like bosons (rather than fermions) because we are not counting explicitly the color quantum number. The wavefunction of the system is completely anti-symmetric in color (to make it color invariant) so that the Pauli principle requires it to be symmetric in the remaining quantum numbers. The null momentum of each parton is positive so each has to be less than the total momentum P . The main effect of a finite (but large) N_c is thus to require that $p < P$: $\tilde{\psi}(p_1, \dots, p_N) = 0$, if $p_i > P$ which is in addition to the condition that $p_i > 0$. This way we derive exactly the model of interacting partons [13] as an approximation to two-dimensional QCD. As noted in that paper, we can make a mean field approximation keeping this condition in place to take into account of the effect of finite N_c .

Thus we can determine the valence parton distribution function if we can minimize the above energy functional $E_1(\psi)$ subject to the conditions that

$$\tilde{\psi}(p) = 0 \text{ for } p < 0 \text{ and } p > P, \quad \int_0^P |\tilde{\psi}(p)|^2 \frac{dp}{2\pi} = 1, \quad \int_0^P p |\tilde{\psi}(p)|^2 \frac{dp}{2\pi} = fP. \quad (8)$$

Here f is the fraction of the momentum carried by the valence partons.

This problem has been solved numerically [13] as well as in a variational approximation [4]. The results can then be compared to the experimental measurements of the $x F_3$ structure function. The agreement is quite good, confirming our picture of the structure of a hadron [4]. In particular we have resolved the apparent difference between the Skyrme model of the baryon and the valence parton model: we have found a topological soliton model that applies to high energy scattering from which an interacting valence parton model can be derived. We can derive more information such as

spin and flavor-dependent or anti-quark distributions functions [12], and we hope to address these issues in later papers.

Acknowledgement: I thank V. John and G. S. Krishnaswami for discussions. This research was supported in part by the US DOE grant DE-FG02-91ER40685.

- [1] J.D. Bjorken, *Partons*, Invited talk Presented at the Int. Conf. on Duality and Symmetry in Hadron Physics, Tel-Aviv, 1971; R. P. Feynman, *Photon-Hadron Interactions* Benjamin, Reading (1972).
- [2] L. V. Gribov and L. N. Lipatov, *Yad. Fiz.* 15, 781 [*Sov. J.Nucl.Phys.*15, 438] (1972); G. Altarelli and G. Parisi, *Nucl. Phys.* B126, 298 (1977); Yu. L. Dokshitzer, *Sov. Phys. JETP* 46(4) (1977).
- [3] See e.g., M. Gluck and E. Reya, *Nucl. Phys.* **B130**, 76 (1977); A.D. Martin, R.G. Roberts, W.J. Stirling and R.S. Thorne, *Eur. Phys. J.* **C4**, 463 (1998).
- [4] G. S. Krishnaswami and S. G. Rajeev, *The x-dependence of Parton Distributions Compared with Neutrino Data*, in preparation.
- [5] S.J. Brodsky, hep-ph/9604391; M. Diehl, T. Feldmann, R. Jakob and P. Kroll, *Eur. Phys. J.* **C8**, 409 (1999) hep-ph/9811253.
- [6] S.G. Rajeev, ‘In 1991 Summer School on High Energy Physics and Cosmology’, ed. E. Gava, et. al. World Scientific, Singapore,(1992); P.F. Bedaque, I. Horvath, S.G. Rajeev, *Mod.Phys.Lett.A*7:3347-3356,1992;hep-th/9209027;S.G. Rajeev, *Int.J.Mod.Phys.A*9:5583,1994,hep-th/9401115;S.G. Rajeev and O.T. Turgut, *Comm. Math. Phys.* 192, 493-517, (1998), hep-th/9705103.
- [7] A.J. Buras, *Rev. Mod. Phys.* **52**, 199 (1980); Y.L. Dokshitzer, V.A. Khoze, A.H. Mueller and S.I. Troian, “Basics of perturbative QCD,” *Gif-sur-Yvette, France: Ed. Frontieres (1991)*; G. Sterman *et al.*, *Rev. Mod. Phys.*, Vol 67, No 1. (1995).
- [8] S. G. Rajeev, *Derivation of the Hadronic Structure Functions from Quantum Chromodynamics*, Lectures at the Swedish Academy of Sciences and the Feza Gursey Institute, to appear in *Frontiers in Physics* ed. D. Pines: hep-th/9905072.
- [9] A. Pressley and G. Segal, *Loop Groups* Clarendon Press, Oxford (1986).
- [10] T. H. R. Skyrme, *Proc. Royal. Soc. Lond.* A260,127(1961);*Nucl. Phys.* 31, 556(1961); *J. Math. Phys.* 12, 556(1962); A. P. Balachandran, V. P. Nair, S. G. Rajeev and A. Stern, *Phys. Rev. Lett.*, 49,1124 (1982); *Phys. Rev.* D27,1153 (1983); S. G. Rajeev, *Phys. Rev.* D29, 2844 (1984);E. Witten, *Nucl. Phys.* B223, 422 (1983); B223, 433 (1983); G. S. Adkins, C. R. Nappi and E. Witten, *Nucl. Phys.* B228, 552 (1983).
- [11] K. S. Gupta, S. Guruswamy and S. G. Rajeev, *Phys. Rev.* D48, 3354 (1993),hep-ph/9301208
- [12] V. John, G. S. Krishnaswami and S. G. Rajeev, *The Anti-quark Distribution function of the Baryon*, in preparation.
- [13] G. S. Krishnaswami and S. G. Rajeev, *Phys. Lett.* B441, 449 (1998).